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Deformed field theory on κ-spacetime

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Abstract. A general formalism is developed that allows the construction of a field theory on quantum spaces which are deformations of ordinary spacetime. The symmetry group of spacetime (the Poincaré group) is replaced by a quantum group. This formalism is demonstrated for the κ-deformed Poincaré algebra and its quantum space. The algebraic setting is mapped to the algebra of functions of commuting variables with a suitable $\star$-product. Fields are elements of this function algebra. The Dirac and Klein–Gordon equation are defined and an action is found from which they can be derived.

1 Introduction

All experimental evidence supports the assumption that spacetime forms a differential manifold. All successful fundamental theories are formulated as field theories on such manifolds.

Nevertheless, in quantum field theories (QFT) we meet some intrinsic difficulties at very high energies or very short distances that do not seem to be resolvable in the framework of QFT. It seems that the structure of QFT has to be modified somewhere. We have no hints from experiments where and how this should be done.

In a very early attempt – almost at the beginning of QFT – it was suggested by Heisenberg [1] that spacetime might be modified at very short distances by algebraic properties that could lead to uncertainty relations for the space coordinates.

This idea was worked out by Snyder [2] in a specific model. He gave a very systematic analysis and physical interpretation of such a structure. Pauli, in a letter to Bohr [3] called it "a mathematically ingenious proposal, which, however, seems to be a failure for reasons of physics".

In the meantime experimental data for physics at much shorter distances have become available. At the same time mathematical methods have improved enormously and it seems to be time to exploit the idea again.

In mathematics the concept of "deformation" has shown to be extremely fruitful. Especially the deformation of groups to Hopf algebras [4–7], the so-called quantum groups, has opened a new field in mathematics. At the same time the deformation of quantum mechanics [8] has seen a very exciting development as well.

In this paper we try to bring these two concepts together aiming at a deformed field theory (DFT). It is not a differential manifold on which we formulate such a theory; it is rather formulated on a quantum space.

An example is the canonical quantum space, where the coordinates $\hat{x}^\mu$ are subject to the relations

$$[\hat{x}^\mu, \hat{x}^\nu] = \theta^{\mu\nu},$$

with constant $\theta^{\mu\nu}$. This structure has been investigated in many papers (see e.g. [9] and the references in [10, 11]). There is, however, no quantum group associated with this quantum space.

We expect additional features of a field theory from a quantum group that can be interpreted as a deformation of the Poincaré group. The simplest example is the κ-deformed Poincaré algebra and its associated quantum space. In this paper we treat the Euclidean version:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu), \qquad \mu = 1, \ldots n.$$

The algebraic structure of the κ-deformed Poincaré algebra has been investigated intensively; see e.g. [12–15].

In this paper we have systematically developed the approach starting from the coordinate algebra in the spirit of Manin's discussion of $SU_q(2)$ [16]. In this approach, the coordinate algebra becomes a factor space and all maps on this space have to respect the factorization property. We then use the isomorphism of the abstract algebra with the

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algebra of functions of commuting variables equipped with a $\star$-product. In a series of papers by Lukierski et al. [17–19] (see also [20]), this model has been treated with the methods of deformation quantization as well. Their work is very similar to our approach.

In Sect. 2 we concentrate on the κ-deformed quantum space in the algebraic setting. We define derivatives, generators of the deformed symmetry algebra, as well as Dirac and Laplace operators, constructing the model systematically on the basis of the quantum space. All formulae are worked out in full detail.

In Sect. 3 we map the algebraic setting into the framework of deformation quantization, introducing a suitable $\star$-product.

In Sect. 4 we introduce fields that are going to be the objects in a DFT. The Klein–Gordon equation and the Dirac equation are formulated.

In Sect. 5 we introduce an integral for an action. Field equations can then be derived by means of a variational principle.

2 The algebra

The symmetry structure of κ-Minkowski spacetime is an example of a quantum group (Hopf algebra) that acts on a quantum space (module). Our aim is to construct quantum field theories with the methods of deformation quantization on such a quantum space and to study the implications of a quantum group symmetry on these field theories.

For this purpose we start from the quantum space and the relations that define it.

Coordinate space

The coordinate space will be the factor space of the algebra freely generated by the coordinates $\hat{x}^1 \dots \hat{x}^n$, divided by the ideal generated by commutation relations [16, 21, 22]. For the κ-Minkowski space the relations are of the Lie algebra-type[1]

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu), \quad \mu = 1, \dots, n. \tag{1}$$

The real parameters a^μ play the role of structure constants for the Lie algebra:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}C^{\mu\nu}_\rho \hat{x}^\rho, \qquad C^{\mu\nu}_\rho = a^\mu \delta^\nu_\rho - a^\nu \delta^\mu_\rho. \tag{2}$$

We here study the Euclidean version. The generalization to a Poincaré version is straightforward, with the direction n (see (3)) either space-, time- or light-like[2]. In the Euclidean case a^μ can be transformed by a linear transformation of the coordinates (rotation) to the form

$$a^\mu = \delta^{\mu n} a. \tag{3}$$

[1] The deformation parameter a^μ is related to the more common κ through $\sqrt{a^2} = \kappa^{-1}$

[2] Compare [19, 23, 24]

The vector a^μ points into the n direction. In this form it is easier to analyse the relations (1); they are

$$\begin{aligned} \left[\hat{x}^i, \hat{x}^j\right] &= 0, \\ \left[\hat{x}^n, \hat{x}^i\right] &= \mathrm{i}a\hat{x}^i, \quad i, j = 1, \dots n-1. \end{aligned} \tag{4}$$

$SO_a(n)$ rotations

A map of the coordinate space has to respect the factor space structure, or, as we say, it has to be consistent with the relations (cf. [16, 21]). Generators of such maps are

$$\begin{aligned} \left[M^{rs}, \hat{x}^i\right] &= \delta^{ri}\hat{x}^s - \delta^{si}\hat{x}^r, \\ [M^{rs}, \hat{x}^n] &= 0, \\ \left[N^l, \hat{x}^i\right] &= -\delta^{li}\hat{x}^n - \mathrm{i}aM^{li}, \\ \left[N^l, \hat{x}^n\right] &= \hat{x}^l + \mathrm{i}aN^l. \end{aligned} \tag{5}$$

We shall call M^{rs} and $N^l = M^{nl}$ generators of $SO_a(n)$, because for $a = 0$ we find the generators of the rotation group $SO(n)$. For $a \neq 0$ we have to check the consistency of (4) and (5). Since this type of calculations will appear again and again, we exhibit an example. We calculate

$$N^l \left(\left[\hat{x}^n, \hat{x}^i\right] - \mathrm{i}a\hat{x}^i \right) \tag{6}$$

term by term using (5):

$$\begin{aligned} N^l \hat{x}^n \hat{x}^i &= \hat{x}^l \hat{x}^i - \mathrm{i}a(\delta^{il}\hat{x}^n + \mathrm{i}aM^{li}) + \mathrm{i}a\hat{x}^i N^l \\ &\quad -\delta^{il}\hat{x}^n\hat{x}^n - \mathrm{i}a\hat{x}^n M^{li} + \hat{x}^n\hat{x}^i N^l, \\ N^l \hat{x}^i \hat{x}^n &= -\delta^{il}\hat{x}^n\hat{x}^n - \mathrm{i}a\hat{x}^n M^{li} + \hat{x}^i\hat{x}^l \\ &\quad +\mathrm{i}a\hat{x}^i N^l + \hat{x}^i\hat{x}^n N^l, \\ N^l(-\mathrm{i}a\hat{x}^i) &= \mathrm{i}a(\delta^{il}\hat{x}^n + \mathrm{i}aM^{li} - \hat{x}^i N^l). \end{aligned}$$

Adding all this we find

$$N^l \left(\left[\hat{x}^n, \hat{x}^i\right] - \mathrm{i}a\hat{x}^i \right) = \left(\left[\hat{x}^n, \hat{x}^i\right] - \mathrm{i}a\hat{x}^i \right) N^l. \tag{7}$$

The consistency of the N^l operations with (one of) the relations (4) is verified.

If we now define the map

$$\hat{x}'^\mu = \hat{x}^\mu + \epsilon_l \left(N^l \hat{x}^\mu \right), \tag{8}$$

we find to first order in ϵ

$$\left[\hat{x}'^n, \hat{x}'^i\right] = \mathrm{i}a\hat{x}'^i, \qquad \left[\hat{x}'^i, \hat{x}'^j\right] = 0. \tag{9}$$

The algebra generated by the rotations is a deformation of the Lie algebra $SO(n)$; we shall call it $SO_a(n)$.

From (5) it is possible to compute the commutators of the generators. As a possible solution (this was considered specifically in [14]) we find the undeformed $SO(n)$ algebra

$$\left[N^l, N^k\right] = M^{lk},$$

$$\left[M^{rs}, N^l\right] = \delta^{rl} N^s - \delta^{sl} N^r, \tag{10}$$
$$\left[M^{rs}, M^{kl}\right] = \delta^{sl} M^{rk} + \delta^{rk} M^{sl} - \delta^{rl} M^{sk} - \delta^{sk} M^{rl}.$$

But the comultiplication will turn out to be quite different when the generators act on functions of $\hat{x}$. This is already apparent from (5). An explicit expression for the comultiplication will contain derivatives as well. Thus, we define derivatives next.

Derivatives

Derivatives on an algebra have been introduced in [21]. They generate a map in the coordinate space – elements of the coordinate space are mapped to other elements of the coordinate space. Thus, they have to be consistent with the algebra relations. For $a = 0$ they should behave like ordinary derivatives, for $a \neq 0$ the Leibniz rule has to be generalized to achieve consistency [21].

We also demand that the derivatives form a module for $SO_a(n)$. In addition they should act at most linearly in the coordinates and the derivatives. These requirements are satisfied by the following rules for differentiation:

$$\left[\hat{\partial}_n, \hat{x}^i\right] = 0,$$
$$\left[\hat{\partial}_n, \hat{x}^n\right] = 1,$$
$$\left[\hat{\partial}_i, \hat{x}^j\right] = \delta_i^j, \tag{11}$$
$$\left[\hat{\partial}_i, \hat{x}^n\right] = \mathrm{i}a\hat{\partial}_i$$

and

$$\left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = 0. \tag{12}$$

The requirement of linearity has been added in order to get an (almost) unique solution [25, 26, 35]. It is not essential for the definition of derivatives.

We can apply derivatives to a function of $\hat{x}^\mu$ and take the derivatives to the right hand side of this function using (11). For the $\hat{\partial}_n$ this yields the usual Leibniz rule; for the $\hat{\partial}_i$ we find that $\hat{x}^n$ is shifted by $\mathrm{i}a$. This can be expressed by the shift operator $\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}$. Note that $\hat{\partial}_n$ commutes with $\hat{x}^i$.

We obtain the Leibniz rule:

$$\hat{\partial}_n(\hat{f} \cdot \hat{g}) = (\hat{\partial}_n \hat{f}) \cdot \hat{g} + \hat{f} \cdot \hat{\partial}_n \hat{g},$$
$$\hat{\partial}_i(\hat{f} \cdot \hat{g}) = (\hat{\partial}_i \hat{f}) \cdot \hat{g} + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n} \hat{f}) \cdot \hat{\partial}_i \hat{g}. \tag{13}$$

Next we construct commutators of the generators of $SO_a(n)$ with the derivatives such that these form a module. For this purpose we perform a power series expansion in a. At lowest order we start from a vector-like behavior of $\hat{\partial}_\mu$. In first order in a we have to modify the commutator to be consistent with (10) and (11). This procedure has to be repeated; finally we find

$$\left[M^{rs}, \hat{\partial}_i\right] = \delta_i^r \hat{\partial}_s - \delta_i^s \hat{\partial}_r,$$
$$\left[M^{rs}, \hat{\partial}_n\right] = 0,$$
$$\left[N^l, \hat{\partial}_i\right] = \delta_i^l \frac{1 - \mathrm{e}^{2\mathrm{i}a\hat{\partial}_n}}{2\mathrm{i}a} - \frac{\mathrm{i}a}{2} \delta_i^l \hat{\Delta} + \mathrm{i}a\hat{\partial}_l \hat{\partial}_i, \tag{14}$$
$$\left[N^l, \hat{\partial}_n\right] = \hat{\partial}_l,$$

where $\hat{\Delta} = \sum_{i=1}^{n-1} \hat{\partial}_i \hat{\partial}_i$. Equations (14) are valid to all orders in a. By a direct construction we have shown that the derivatives form a module of $SO_a(n)$.

We can apply M^{rs} and N^l to a function of $\hat{x}^\mu$ and take the generators to the right hand side. From the result of this calculation we can abstract the comultiplication rule:

$$\Delta N^l = N^l \otimes \mathbf{1} + \mathrm{e}^{\mathrm{i}a\hat{\partial}_n} \otimes N^l - \mathrm{i}a\hat{\partial}_b \otimes M^{lb},$$
$$\Delta M^{rs} = M^{rs} \otimes \mathbf{1} + \mathbf{1} \otimes M^{rs},$$
$$\Delta \hat{\partial}_n = \hat{\partial}_n \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\partial}_n, \tag{15}$$
$$\Delta \hat{\partial}_i = \hat{\partial}_i \otimes \mathbf{1} + \mathrm{e}^{\mathrm{i}a\hat{\partial}_n} \otimes \hat{\partial}_i.$$

This is the comultiplication consistent with the algebra (10). It is certainly different from the comultiplication rule for $SO(n)$.

The comultiplication involves the derivatives. For representations of the algebra (10), where $\hat{\partial}_\mu$ acting on the representation gives zero, the standard comultiplication rule for $SO(n)$ emerges.

As far as the commutators of M^{rs} and N^l with the coordinates and the derivatives are concerned, we can express M^{rs} and N^l by the coordinates and the derivatives, as is usually done for angular momentum:

$$\hat{M}^{rs} = \hat{x}^s \hat{\partial}_r - \hat{x}^r \hat{\partial}_s,$$
$$\hat{N}^l = \hat{x}^l \frac{\mathrm{e}^{2\mathrm{i}a\hat{\partial}_n} - 1}{2\mathrm{i}a} - \hat{x}^n \hat{\partial}_l + \frac{\mathrm{i}a}{2} \hat{x}^l \hat{\Delta}. \tag{16}$$

According to (15), it is natural to consider the generators M^{rs}, N^l and $\hat{\partial}_\mu$ as generators of the a-Euclidean Hopf algebra. It should be noted that the deformed generators M^{rs}, N^l do not form a Hopf algebra by themselves. In the coproduct the derivatives, or equivalently the translations in the a-Euclidean Hopf algebra, appear as well.

Laplace and Dirac operators

A deformed Laplace operator [12,13] and a deformed Dirac operator [29,30] can be defined. For the Laplace operator $\hat{\Box}$ we demand that it commutes with the generators of the a-Euclidean Hopf algebra,

$$\left[M^{rs}, \hat{\Box}\right] = 0, \qquad \left[N^l, \hat{\Box}\right] = 0, \tag{17}$$

and that it is a deformation of the usual Laplace operator. By iteration in a we find[3]:

$$\hat{\Box} = \mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{\Delta} + \frac{2}{a^2}(1-\cos(a\hat{\partial}_n)). \tag{18}$$

Since the γ-matrices are $\hat{x}$-independent and transform as usual, the covariance of the full Dirac operator $\gamma^\mu\hat{D}_\mu$ implies that the transformation law of its components is vector-like:

$$\begin{aligned}
\left[M^{rs},\hat{D}_n\right] &= 0,\\
\left[M^{rs},\hat{D}_i\right] &= \delta_i^r\hat{D}_s - \delta_i^s\hat{D}_r,\\
\left[N^l,\hat{D}_n\right] &= \hat{D}_l,\\
\left[N^l,\hat{D}_i\right] &= -\delta_i^l\hat{D}_n.
\end{aligned} \tag{19}$$

These relations are obviously consistent with the algebra (10). A differential operator that satisfies (19) and that has the correct limit for $a\to 0$ is

$$\begin{aligned}
\hat{D}_n &= \frac{1}{a}\sin\left(a\hat{\partial}_n\right) + \frac{\mathrm{i}a}{2}\hat{\Delta}\,\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n},\\
\hat{D}_i &= \hat{\partial}_i\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n},
\end{aligned} \tag{20}$$

where the derivatives $\hat{\partial}_\mu$ transform according to (14).

The square of the Dirac operator turns out to be (compare [29])

$$\gamma^\mu\hat{D}_\mu\gamma^\nu\hat{D}_\nu = \sum_{\mu=1}^{n}\hat{D}_\mu\hat{D}_\mu = \hat{\Box}\left(1-\frac{a^2}{4}\hat{\Box}\right). \tag{21}$$

Using this we can express the Laplace operator as a function of the Dirac operator:

$$\hat{\Box} = \frac{2}{a^2}\left(1-\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right). \tag{22}$$

The sign of the square root is determined by the limit $a\to 0$. We have dropped the summation symbol.

Dirac operator as a derivative

The Dirac operator $\hat{D}_\mu$ can be seen as a derivative operator as well, having very simple transformation properties under $SO_a(n)$, but as we shall see with a highly non-linear Leibniz rule.

We invert (20) in order to express the derivative operator $\hat{\partial}_\mu$ in terms of the Dirac operator and proceed as follows:

$$\begin{aligned}
\hat{\partial}_i &= \hat{D}_i\mathrm{e}^{\mathrm{i}a\hat{\partial}_n},\\
\hat{\partial}_i\hat{\partial}_i &= \hat{\Delta} = \hat{D}_i\hat{D}_i\mathrm{e}^{2\mathrm{i}a\hat{\partial}_n},
\end{aligned}$$

[3] In this form the Laplace operator has been given in [15]

$$\hat{D}_n = \frac{1}{2\mathrm{i}a}(\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}-\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}) + \frac{\mathrm{i}a}{2}\hat{D}_i\hat{D}_i\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}. \tag{23}$$

Multiplying (23) by $\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}$ leads to a quadratic equation for $\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}$:

$$\mathrm{e}^{-2\mathrm{i}a\hat{\partial}_n} + 2\mathrm{i}a\hat{D}_n\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} + a^2\hat{D}_i\hat{D}_i - 1 = 0.$$

Solving this quadratic equation we find (compare [32])

$$\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} = -\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}. \tag{24}$$

The sign of the square root is again determined by the limit $a\to 0$.

With (22) we find a form that is easier to handle:

$$\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} = 1-\mathrm{i}a\hat{D}_n - \frac{a^2}{2}\hat{\Box}. \tag{25}$$

Multiplying (23) by $\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}$ we find a quadratic equation for $\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}$ with the solution

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}a\hat{\partial}_n} &= \frac{1}{1-a^2\hat{D}_k\hat{D}_k}\left(\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)\\
&= \frac{1}{1-a^2\hat{D}_k\hat{D}_k}\left(1+\mathrm{i}a\hat{D}_n - \frac{a^2}{2}\hat{\Box}\right).
\end{aligned} \tag{26}$$

It is easy to verify that (26) is the inverse of (24).

Now we can invert (20):

$$\begin{aligned}
\hat{\partial}_i &= \frac{\hat{D}_i}{1-a^2\hat{D}_k\hat{D}_k}\left(\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)\\
&= \frac{\hat{D}_i}{1-a^2\hat{D}_k\hat{D}_k}\left(1+\mathrm{i}a\hat{D}_n - \frac{a^2}{2}\hat{\Box}\right),\\
\hat{\partial}_n &= -\frac{1}{\mathrm{i}a}\ln\left(-\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)\\
&= -\frac{1}{\mathrm{i}a}\ln\left(1-\mathrm{i}a\hat{D}_n - \frac{a^2}{2}\hat{\Box}\right).
\end{aligned} \tag{27}$$

To compute the commutator of $\hat{D}_\mu$ and $\hat{x}^\nu$, we use the representation (20), apply (11) and finally express $\hat{\partial}_\mu$ again by $\hat{D}_\mu$ using (27). The result is

$$\begin{aligned}
\left[\hat{D}_n,\hat{x}^j\right] &= \mathrm{i}a\hat{D}_j,\\
\left[\hat{D}_n,\hat{x}^n\right] &= \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu},\\
\left[\hat{D}_i,\hat{x}^j\right] &= \delta_i^j\left(-\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right),\\
\left[\hat{D}_i,\hat{x}^n\right] &= 0.
\end{aligned} \tag{28}$$

For two functions $\hat{f}(\hat{x})$ and $\hat{g}(\hat{x})$ the Leibniz rule can be computed from (28):

$$\hat{D}_n(\hat{f}\cdot\hat{g}) = (\hat{D}_n\hat{f})\cdot(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{g}) + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{D}_n\hat{g})$$

$$+\mathrm{i}a(\hat{D}_j e^{\mathrm{i}a\hat{\partial}_n}\hat{f}) \cdot (\hat{D}_j\hat{g}),$$
$$\hat{D}_i(\hat{f}\cdot\hat{g}) = (\hat{D}_i\hat{f})\cdot(e^{-\mathrm{i}a\hat{\partial}_n}\hat{g}) + \hat{f}\cdot(\hat{D}_i\hat{g}). \quad (29)$$

For $e^{\pm\mathrm{i}a\hat{\partial}_n}$ the expressions (24) and (26) have to be inserted.

Equation (20) tells us that the Dirac operator $\hat{D}_\mu$ is in the enveloping algebra of $\hat{\partial}_\mu$ and (27) that the derivative operator $\hat{\partial}_\mu$ is in the enveloping algebra of $\hat{D}_\mu$. Equations (20) and (27) can be interpreted as a change of basis in the derivative algebra (compare [32]).

One basis $\{\hat{D}_\mu\}$ has simple transformation properties; the other basis $\{\hat{\partial}_\mu\}$ has a simple Leibniz rule.

The a-Euclidean Hopf algebra might also be generated by M^{rs}, N^l and $\hat{D}_\mu$. This will be of advantage if we focus on the $SO_a(n)$ behavior:

$$\begin{aligned}
[M^{rs}, M^{tu}] &= \delta^{rt}M^{su} + \delta^{su}M^{rt} - \delta^{st}M^{ru} - \delta^{ru}M^{st},\\
\left[M^{rs}, N^l\right] &= \delta^{rl}N^s - \delta^{sl}N^r,\\
\left[N^k, N^l\right] &= M^{kl}, \qquad (30)\\
\left[M^{rs}, \hat{D}_n\right] &= 0,\\
\left[M^{rs}, \hat{D}_i\right] &= \delta^{ri}\hat{D}_s - \delta^{si}\hat{D}_r,\\
\left[N^l, \hat{D}_n\right] &= \hat{D}_l,\\
\left[N^l, \hat{D}_i\right] &= -\delta^{li}\hat{D}_n,\\
\left[\hat{D}_\mu, \hat{D}_\nu\right] &= 0.
\end{aligned}$$

This again is the undeformed algebra – a does not appear. Of course, the comultiplication in this basis depends on a:

$$\begin{aligned}
\Delta M^{rs} &= M^{rs}\otimes \mathbf{1} + \mathbf{1}\otimes M^{rs},\\
\Delta N^l &= N^l \otimes \mathbf{1} + \frac{\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_j\hat{D}_j}\otimes N^l\\
&\quad -\frac{\mathrm{i}a\hat{D}_k}{1-a^2\hat{D}_j\hat{D}_j}\left(\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)\otimes M^{lk},\\
\Delta\hat{D}_n &= \hat{D}_n\otimes\left(-\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)\\
&\quad + \frac{\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_j\hat{D}_j}\otimes\hat{D}_n \qquad (31)\\
&\quad + \mathrm{i}a\frac{\hat{D}_k}{1-a^2\hat{D}_j\hat{D}_j}\left(\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)\otimes\hat{D}_k,\\
\Delta\hat{D}_i &= \hat{D}_i\otimes\left(-\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right) + \mathbf{1}\otimes\hat{D}_i.
\end{aligned}$$

To define a Hopf algebra, multiplication and comultiplication are essential[4].

Conjugation

All the relations that we have considered do not change under the formal involution that we shall call conjugation:

$$(\hat{x}^\mu)^+ = \hat{x}^\mu, \qquad (\hat{\partial}_\mu)^+ = -\hat{\partial}_\mu,$$
$$(M^{rs})^+ = -M^{rs}, \qquad (N^l)^+ = -N^l. \qquad (32)$$

The order of algebraic elements in the product has to be inverted under conjugation.

It is easy to show that $\hat{N}^l$ and $\hat{M}^{rs}$ in (16) have the desired conjugation property by conjugating $\hat{x}^\mu$ and $\hat{\partial}_\mu$.

3 The ⋆-product

Our aim is to formulate a field theory on the algebra discussed in the previous section with the methods of deformation quantization. The algebraic formalism is connected with deformation quantization via the ⋆-product. The idea in short is as follows. We consider polynomials of fixed degree in the algebra – homogeneous polynomials. They form a finite-dimensional vector space. If the algebra has the Poincaré–Birkhoff–Witt property, and all Lie algebras have this property, then the dimension of the vector space of homogeneous polynomials in the algebra is the same as for polynomials of commuting variables. Thus, there is an isomorphism between the two finite-dimensional vector spaces. This vector space isomorphism can be extended to an algebra isomorphism by defining the product of polynomials of commuting variables by first mapping these polynomials back to the algebra, multiplying them there and mapping the product to the space of polynomials of ordinary variables. The product we obtain that way is called a ⋆-product. It is non-commutative and contains the information about the product in the algebra. The objects that we will identify with physical fields are functions. This is possible because the ⋆-product of polynomials can be extended to the ⋆-product of functions.

The ⋆-product for Lie algebras

There is a standard ⋆-product for Lie algebras [34]. If $\hat{x}^\mu$ are the generators of a Lie algebra such that

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}C^{\mu\nu}_\rho\hat{x}^\rho, \qquad (33)$$

then a ⋆-product can be computed with the help of the Baker–Campbell–Hausdorff formula:

$$e^{\mathrm{i}\hat{x}^\nu p_\nu}e^{\mathrm{i}\hat{x}^\nu q_\nu} = e^{\mathrm{i}\hat{x}^\nu\{p_\nu+q_\nu+\frac{1}{2}g_\nu(p,q)\}}. \qquad (34)$$

[4] The additional ingredients for a Hopf algebra, counit and antipode, have been calculated, e.g. [27]

The exponential, when expanded, is always fully symmetric in the algebraic elements $\hat{x}^\nu$. Therefore we call this $\star$-product the symmetric $\star$-product. In the following we shall use the symmetric $\star$-product. It is

$$f \star g(z) = \lim_{\substack{x \to z \\ y \to z}} \exp\left(\frac{\mathrm{i}}{2} z^\nu g_\nu(\mathrm{i}\partial_x, \mathrm{i}\partial_y)\right) f(x)g(y). \quad (35)$$

It can be applied to any Lie algebra, but in general there is no closed form for $g_\nu(\mathrm{i}\partial_x, \mathrm{i}\partial_y)$. It can, however, be computed in a power series expansion in the structure constants $C_\rho^{\mu\nu}$. We obtain

$$[x^\mu \stackrel{\star}{,} x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}C_\rho^{\mu\nu} x^\rho. \quad (36)$$

All the consequences of the algebraic relation (33) can be derived from the $\star$-product.

If the algebra allows for a conjugation, then the symmetric $\star$-product has the property

$$\overline{f \star g} = \bar{g} \star \bar{f}. \quad (37)$$

The bar denotes complex conjugation.

We have found a closed form for the symmetric $\star$-product for the algebra (4) [35] (compare [18]). Using the abbreviations

$$\partial_{x^n} = \frac{\partial}{\partial x^n}, \quad \partial_{y^n} = \frac{\partial}{\partial y^n}, \quad \partial_n = \frac{\partial}{\partial x^n} + \frac{\partial}{\partial y^n}, \quad (38)$$

the $\star$-product takes the form

$$\begin{aligned} &f \star g(z) \\ &= \lim_{\substack{x \to z \\ y \to z}} \exp\left(z^j \partial_{x^j}\left(\frac{\partial_n}{\partial_{x^n}} \mathrm{e}^{-\mathrm{i}a\partial_{y^n}} \frac{1 - \mathrm{e}^{-\mathrm{i}a\partial_{x^n}}}{1 - \mathrm{e}^{-\mathrm{i}a\partial_n}} - 1\right)\right. \\ &\quad \left. + z^j \partial_{y^j}\left(\frac{\partial_n}{\partial_{y^n}} \frac{1 - \mathrm{e}^{-\mathrm{i}a\partial_{y^n}}}{1 - \mathrm{e}^{-\mathrm{i}a\partial_n}} - 1\right)\right) f(x)g(y). \end{aligned} \quad (39)$$

To second order in a we obtain

$$\begin{aligned} f \star g\,(x) &= f(x)g(x) + \frac{\mathrm{i}a}{2} x^j (\partial_n f(x) \partial_j g(x) \\ &-\partial_j f(x) \partial_n g(x)) - \frac{a^2}{12} x^j (\partial_n^2 f(x) \partial_j g(x) \\ &-\partial_j \partial_n f(x) \partial_n g(x) - \partial_n f(x) \partial_j \partial_n g(x) + \partial_j f(x) \partial_n^2 g(x)) \\ &- \frac{a^2}{8} x^j x^k (\partial_n^2 f(x) \partial_j \partial_k g(x) - 2\partial_j \partial_n f(x) \partial_n \partial_k g(x) \\ &+ \partial_j \partial_k f(x) \partial_n^2 g(x)) + \mathcal{O}(a^3). \end{aligned} \quad (40)$$

The a-Euclidean Hopf algebra and the $\star$-product

The operators $\hat{\partial}_\mu$, M^{rs} and N^l generate transformations on the coordinate space. In a standard way maps in the coordinate space can be mapped to maps of the space of functions of commuting variables.

We first consider the derivatives

$$\hat{\partial}_\mu \to \partial^*_\mu, \quad (41)$$

where ∂^*_μ is the image of the algebraic map $\hat{\partial}_\mu$, and as such it is a map of the space of functions of commuting variables into itself. In the following, the derivatives ∂_μ will always be the ordinary derivatives $\frac{\partial}{\partial x^\mu}$ on functions of commuting variables. Such mappings have previously been discussed in [13,17,28].

From the action of $\hat{\partial}_\mu$ on symmetric polynomials we can compute the action of ∂^*_μ on ordinary functions[5]:

$$\begin{aligned} \partial^*_n f(x) &= \partial_n f(x), \\ \partial^*_i f(x) &= \partial_i \frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{\mathrm{i}a\partial_n} f(x). \end{aligned} \quad (42)$$

The derivatives have inherited the Leibniz rule (13):

$$\begin{aligned} \partial^*_n(f \star g(x)) &= (\partial^*_n f(x)) \star g(x) + f(x) * (\partial^*_n g(x)), \\ \partial^*_i(f \star g(x)) &= (\partial^*_i f(x)) \star g(x) \\ &\quad + (\mathrm{e}^{\mathrm{i}a\partial^*_n} f(x)) \star (\partial^*_i g(x)). \end{aligned} \quad (43)$$

We proceed in an analogous way for the generators M^{rs} and N^l. The result is (cf. the momentum representation in e.g. [13,19])

$$\begin{aligned} N^{*l} f(x) &= \left(x^l \partial_n - x^n \partial_l + x^l \partial_\mu \partial_\mu \frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{2\partial_n}\right. \\ &\quad \left. - x^\nu \partial_\nu \partial_l \frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1 - \mathrm{i}a\partial_n}{\mathrm{i}a\partial_n^2}\right) f(x), \\ M^{*rs} f(x) &= (x^s \partial_r - x^r \partial_s) f(x). \end{aligned} \quad (44)$$

The comultiplication rule (15) can be reproduced as well:

$$\begin{aligned} &N^{*l}\,(f \star g(x)) \\ &= \Big(N^{*l} f(x)\Big) \star g(x) + \Big(\mathrm{e}^{\mathrm{i}a\partial^*_n} f(x)\Big) \star \Big(N^{*l} g(x)\Big) \\ &\quad -\mathrm{i}a\,(\partial^*_b f(x)) \star \Big(M^{*lb} g(x)\Big), \quad (45) \\ &M^{*rs}\,(f \star g(x)) \\ &= (M^{*rs} f(x)) \star g(x) + f(x) \star (M^{*rs} g(x)). \end{aligned}$$

The algebra of functions with the $\star$-product as multiplication can now be seen as a module for the a-Euclidean Hopf algebra.

In the previous chapter we have seen that the Dirac operator $\hat{D}_\mu$ can be interpreted as a derivative as well. It is natural to carry it over to the algebra of functions with the $\star$-product:

$$D^*_n f(x) = \left(\frac{1}{a} \sin(a\partial_n) + \frac{\Delta_{\mathrm{cl}}}{\mathrm{i}a\partial_n^2}(\cos(a\partial_n) - 1)\right) f(x),$$

[5] In this form first given in [31]

$$D_i^* f(x) = \partial_i \frac{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1}{-\mathrm{i}a\partial_n} f(x), \tag{46}$$

where $\Delta_{\mathrm{cl}} = \partial_i \partial_i$. The Leibniz rule for the Dirac operator is

$$\begin{aligned} D_n^*(f \star g(x)) &= (D_n^* f(x)) \star (\mathrm{e}^{-\mathrm{i}a\partial_n^*} g(x)) \\ &\quad + (\mathrm{e}^{\mathrm{i}a\partial_n^*} f(x)) \star (D_n^* g(x)) \\ &\quad + \mathrm{i}a \left(D_j^* \mathrm{e}^{\mathrm{i}a\partial_n^*} f(x) \right) \star (D_j^* g(x)), \\ D_i^*(f \star g(x)) &= (D_i^* f(x)) \star (\mathrm{e}^{-\mathrm{i}a\partial_n^*} g(x)) \\ &\quad + f(x) \star (D_i^* g(x)). \end{aligned} \tag{47}$$

Finally, for the Laplace operator $\hat{\Box}$ we have

$$(\Box^* f(x)) = -\frac{2}{a^2 \partial_n^2} (\cos(a\partial_n) - 1) \left(\Delta_{\mathrm{cl}} + \partial_n^2 \right) f(x). \tag{48}$$

4 Field equations

Fields

Physical fields are formal power series expansions in the coordinates and as such elements of the coordinate algebra:

$$\hat{\phi}(\hat{x}) = \sum_{\{\alpha\}} c_{\alpha_1 \ldots \alpha_n} \ : (\hat{x}^1)^{\alpha_1} \ldots (\hat{x}^n)^{\alpha_n} : . \tag{49}$$

The summation is over a basis in the coordinate algebra as indicated by the double points. The field can also be defined by its coefficient functions $c_{\{\alpha_1 \ldots \alpha_n\}}$, once the basis is specified.

Fields can be added, multiplied, differentiated and transformed. A transformation is a map in the algebra and as such can be seen as a map of the coefficient functions. We are interested in the maps that are induced by the transformations N_l:

$$\hat{x}'^{\mu} = \hat{x}^{\mu} + \epsilon_l \left(N^l \hat{x}^{\mu} \right). \tag{50}$$

The action of N^l on the coordinates was given in (5). This expresses the transformed coordinate $\hat{x}'$ in terms of the coordinates $\hat{x}$.

The transformation law of a scalar field is defined as usual:

$$\hat{\phi}'(\hat{x}') = \hat{\phi}(\hat{x}). \tag{51}$$

This should be seen as an identity in $\hat{x}$ or $\hat{x}'$. We write it in the given basis:

$$\begin{aligned} \hat{\phi}'(\hat{x}') &= \sum_{\{\alpha\}} c'_{\alpha_1 \ldots \alpha_n} : \left(\hat{x}'^1\right)^{\alpha_1} \ldots \left(\hat{x}'^n\right)^{\alpha_n} : \\ &= \sum_{\{\alpha\}} c'_{\alpha_1 \ldots \alpha_n} : \left(\hat{x}^1 + \epsilon_l \left(N^l \hat{x}^1\right)\right)^{\alpha_1} \ldots \left(\hat{x}^n + \epsilon_l \left(N^l \hat{x}^n\right)\right)^{\alpha_n} : \\ &= \sum_{\{\beta\}} c_{\beta_1 \ldots \beta_n} : \left(\hat{x}^1\right)^{\beta_1} \ldots \left(\hat{x}^n\right)^{\beta_n} : . \end{aligned} \tag{52}$$

This allows us to compute $c_{\beta_1 \ldots \beta_n}$ as a function of $c'_{\alpha_1 \ldots \alpha_n}$ or vice versa.

Spinor fields are defined analogously:

$$\hat{\psi}'_{\sigma}\left(\hat{x}'\right) = \left(1 + \epsilon_l N^l_{\mathrm{rep}}\right)_{\sigma\rho} \hat{\psi}_{\rho}\left(\hat{x}\right), \tag{53}$$

where N^l_{rep} is a representation of N^l acting on coordinate independent spinors. The generalization to vector fields or tensor fields is obvious.

These transformation laws have to be formulated in the $\star$-product language; equation (51) becomes

$$\phi'(x'(x)) = \phi(x), \tag{54}$$

or, transforming the coordinates,

$$\phi'(x) = \phi(x) - \epsilon_l N^{*l} \phi(x), \tag{55}$$

where N^{*l} operates on the coordinates.

The generalization to the operators M^{*rs} and ∂^*_{μ} is straightforward.

Field equations

We introduce the a-deformed Klein–Gordon equation for scalar fields

$$\left(\hat{\Box} + m^2\right) \hat{\phi}(\hat{x}) = 0. \tag{56}$$

The invariance of this equation follows from (17) and (51):

$$\left(\hat{\Box}' + m^2\right) \hat{\phi}'(\hat{x}') = \left(\hat{\Box} + m^2\right) \hat{\phi}(\hat{x}). \tag{57}$$

Similarly the a-deformed Dirac equation

$$\left(\mathrm{i}\gamma^{\lambda} \hat{D}_{\lambda} - m\right) \hat{\psi}(\hat{x}) = 0 \tag{58}$$

is covariant:

$$\left(\mathrm{i}\gamma^{\lambda} \hat{D}'_{\lambda} - m\right) \hat{\psi}'(\hat{x}') = (1 + \epsilon_l N^l_{\mathrm{rep}}) \left(\mathrm{i}\gamma^{\lambda} \hat{D}_{\lambda} - m\right) \hat{\psi}(\hat{x}). \tag{59}$$

These equations take the following form in the $\star$-formalism:

$$\begin{aligned} &\left(\Box^* + m^2\right) \phi(x) \\ &= \left(-\frac{2}{a^2 \partial_n^2} (\cos(a\partial_n) - 1) \left(\Delta_{\mathrm{cl}} + \partial_n^2\right) + m^2 \right) \phi(x) \end{aligned} \tag{60}$$

and

$$\begin{aligned} &\left(\mathrm{i}\gamma^{\lambda} D^*_{\lambda} - m\right) \psi(x) \\ &= \left(\gamma^n \left(\frac{\mathrm{i}}{a} \sin(a\partial_n) + \frac{\Delta_{\mathrm{cl}}}{a\partial_n^2} (\cos(a\partial_n) - 1) \right) \right. \\ &\qquad \left. + \mathrm{i}\gamma^j \partial_j \frac{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1}{-\mathrm{i}a\partial_n} - m \right) \psi(x). \end{aligned} \tag{61}$$

We have defined the a-deformed Klein–Gordon and Dirac equation for fields that are functions of the commuting variables and have to be multiplied with the $\star$-product. These equations are covariant under the a-Euclidean transformations.

5 The variational principle

We will derive field equations by means of a variational principle such that the dynamics can be formulated with the help of the Lagrangian formalism. For this purpose we need an integral. Algebraically an integral is a linear map of the algebra into complex numbers:

$$\int : \hat{\mathcal{A}}(\hat{x}) \longrightarrow \mathbb{C}, \tag{62}$$

$$\int (c_1\hat{f} + c_2\hat{g}) = c_1 \int \hat{f} + c_2 \int \hat{g},$$
$$\forall \hat{f}, \hat{g} \in \hat{\mathcal{A}}(\hat{x}), c_i \in \mathbb{C}. \tag{63}$$

In addition we demand the trace property:

$$\int \hat{f}\hat{g} = \int \hat{g}\hat{f}. \tag{64}$$

In our case this is essential to define the variational principle. To find a workable definition of such an integral we will try to define it in the ⋆-product formalism. There we can use the usual definition of an integral of functions of commuting variables. Such an integral will certainly have the linear property (63), but in general it will not have the trace property (64). It has, however, been shown in [33] that a measure can be introduced to achieve the trace property:

$$\int \mathrm{d}^n x \ \mu(x) \ (f(x) \star g(x)) = \int \mathrm{d}^n x \ \mu(x) \ (g(x) \star f(x)). \tag{65}$$

Note that $\mu(x)$ is not ⋆-multiplied with the other functions; it is part of the volume element.

It turns out that for the ⋆-product (39) and $\mu(x)$ with the property

$$\partial_n \mu(x) = 0, \qquad x^j\partial_j\mu(x) = (1-n)\mu(x), \tag{66}$$

equation (65) will be true. This was shown to first order in a [33], but it can be generalized to the full ⋆-product (39) [35].

Technically $\mu(x)$ is needed because z^j occurs in the exponent of (39). Partially integrating, this z^j has to be differentiated as well. As $\mu(x)$ has the property (66) we find

$$\int \mathrm{d}^n x \ \mu(x) f(x)(x^j\partial_j g(x))$$
$$\to - \int \mathrm{d}^n x \ \mu(x)(x^j\partial_j f(x))g(x). \tag{67}$$

Expanding the exponent in (39) and using (67), (65) can be verified.

An integral with a measure $\mu(x)$ satisfying (66) has the additional property:

$$\int \mathrm{d}^n x \ \mu(x) \ (f(x) \star g(x)) = \int \mathrm{d}^n x \ \mu(x) \ f(x)g(x). \tag{68}$$

For an arbitrary number of functions multiplied with the ⋆-product we can cyclically permute the functions under the integral

$$\int \mathrm{d}^n x \ \mu \ (f_1 \star f_2 \star \cdots \star f_k) = \int \mathrm{d}^n x \ \mu \ (f_k \star f_1 \star f_2 \star \cdots \star f_{k-1}). \tag{69}$$

Thus, any such function can be brought to the left or right hand side of the product. For a variation of some linear combination of such products we always can bring the function to be varied to one side and then vary it:

$$\frac{\delta}{\delta g(x)} \int \mathrm{d}^n x \ \mu \ f \star g \star h = \frac{\delta}{\delta g(x)} \int \mathrm{d}^n x \ \mu \ g(h \star f) = \mu \ h \star f. \tag{70}$$

Hermitean differential operators

We shall call a differential operator $\mathcal{O}$ hermitean if

$$\int \mathrm{d}^n x \ \mu \ \bar{f} \star \mathcal{O}g = \int \mathrm{d}^n x \ \mu \ \overline{\mathcal{O}f} \star g. \tag{71}$$

It is easy to see that the operators $\mathrm{i}\partial_i^*$ or iD_μ^* are not hermitean by this definition of hermiticity, though they are by the algebraic definition (32).

Let us first have a look at the differential operator ∂_i^* as given in (42). Due to the property (66) of μ there is no problem in partially integrating ∂_n. We have

$$\int \mathrm{d}^n x \ \mu \ \bar{f} \star (\partial_i^* g)$$
$$= \int \mathrm{d}^n x \ \mu \ \bar{f} \ (\partial_i^* g) = \int \mathrm{d}^n x \ \mu \ \frac{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1}{-\mathrm{i}a\partial_n} \bar{f} \ \partial_i g$$
$$= - \int \mathrm{d}^n x \ \mu \ \overline{\partial_i^* f} \ g - \int \mathrm{d}^n x \ \partial_i \mu \ \overline{\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{\mathrm{i}a\partial_n} f} \ g. \tag{72}$$

This is quite similar to the case of polar coordinates in ordinary spacetime where $\mathrm{i}\frac{\partial}{\partial r}$ is not hermitean due to the measure $r^2\mathrm{d}r$, but $\mathrm{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$ is hermitean. It is tempting to try a similar strategy here. We first define ρ_i, which is a logarithmic derivative of μ:

$$\rho_i = \frac{\partial_i \mu}{2\mu}. \tag{73}$$

It inherits from μ the following properties:

$$x^l\partial_l\rho_i = -\rho_i \quad \text{and} \quad \partial_n\rho_i = 0. \tag{74}$$

Adding ρ_i to ∂_i renders a derivative $\mathrm{i}\tilde{\partial}_i^*$:

$$\mathrm{i}\tilde{\partial}_i^* = \mathrm{i}(\partial_i + \rho_i)\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{\mathrm{i}a\partial_n}. \tag{75}$$

This $\mathrm{i}\tilde{\partial}_i^*$ is hermitean in the sense of (71):

$$\int \mathrm{d}^n x \ \mu \ \bar{f} \ \ \mathrm{i}(\partial_i + \rho_i)\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{\mathrm{i}a\partial_n} g$$

$$= \int \mathrm{d}^n x\, \mu\, \overline{\mathrm{i}(\partial_i + \rho_i)\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{\mathrm{i}a\partial_n} f}\;\; g. \tag{76}$$

The same strategy works for D^*_μ:

$$D^*_i \longrightarrow \tilde{D}^*_i = (\partial_i + \rho_i)\frac{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1}{-\mathrm{i}a\partial_n},$$

$$D^*_n \longrightarrow \tilde{D}^*_n = \frac{1}{\mathrm{i}a\partial_n^2}(\partial_k + \rho_k)(\partial_k + \rho_k)(\cos(a\partial_n) - 1) + \frac{1}{a}\sin(a\partial_n). \tag{77}$$

These $\mathrm{i}\tilde{D}^*_\mu$ are hermitean in the sense of (71).

The substitution

$$\mathrm{i}\partial_i \longrightarrow \mathrm{i}(\partial_i + \rho_i) =: \pi_i \tag{78}$$

does not change the canonical commutation relations,

$$\left[x^j, x^k\right] = 0, \qquad [\mathrm{i}\partial_i, \mathrm{i}\partial_l] = 0, \qquad \left[\mathrm{i}\partial_i, x^j\right] = \mathrm{i}\delta^j_i, \tag{79}$$

which implies

$$\left[x^j, x^k\right] = 0, \qquad [\pi_i, \pi_l] = 0, \qquad \left[\pi_i, x^j\right] = i\delta^j_i, \tag{80}$$

and vice versa. This can be seen by making use of the properties (74) of ρ_i.

Thus, replacing $\mathrm{i}\partial_i$ by π_i does not change the algebraic properties of the differential operators. This suggests one to introduce $\tilde{M}^{*rs}$ and $\tilde{N}^{*l}$ as well. These operators will satisfy the same commutation relations as M^{*rs}, N^{*l}, ∂^*_μ and D^*_μ. In the sense of (71) the operators $\mathrm{i}\tilde{M}^{*rs}$ will be hermitean, $\mathrm{i}\tilde{N}^{*l}$ not.

A proper action for a spinor field $\tilde{\psi}$ would be

$$S = \int \mathrm{d}^n x\, \mu\, \overline{\tilde{\psi}} \star (\mathrm{i}\gamma^\lambda \tilde{D}^*_\lambda - m)\tilde{\psi}. \tag{81}$$

By varying with respect to $\overline{\tilde{\psi}}$ we obtain

$$\mu(\mathrm{i}\gamma^\lambda \tilde{D}^*_\lambda - m)\tilde{\psi} = 0. \tag{82}$$

Guided by the example of polar coordinates we compute

$$\tilde{D}^*_i \mu^\alpha = \mu^\alpha(\partial_i + (2\alpha + 1)\rho_i)\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{\mathrm{i}a\partial_n}, \tag{83}$$

and similar for $\tilde{D}^*_n$. If we choose $\alpha = -1/2$ we obtain

$$\tilde{D}^*_\lambda \mu^{-\frac{1}{2}} = \mu^{-\frac{1}{2}} D^*_\lambda. \tag{84}$$

This suggests the introduction of the field

$$\tilde{\psi} = \mu^{-\frac{1}{2}}\psi. \tag{85}$$

The field ψ satisfies the Dirac equation as it was introduced in (61),

$$(\mathrm{i}\gamma^\lambda D^*_\lambda - m)\psi = 0. \tag{86}$$

This equation can be derived from the action

$$S = \int \mathrm{d}^n x\, \overline{\psi}(\mathrm{i}\gamma^\lambda D^*_\lambda - m)\psi, \tag{87}$$

which is exactly the action we obtain by substituting $\tilde{\psi} \rightarrow \mu^{-\frac{1}{2}}\psi$ in the action (81), after dropping the $\star$ from the integral with the help of μ.

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